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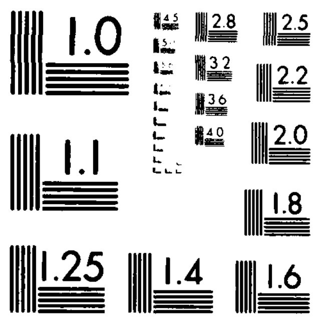

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EULER'S CONTRIBUTION TO CARDINAL SPLINE INTERPOLATION: THE EXPONENTIAL EULER SPLINES

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ABSTRACT

This paper is the author's contribution to the volume "Leonardt Euler-Gedenkband 1983" to be published in Basel, Switzerland, in 1983 in honor of Euler's bicentenial. It is mainly devoted to the exponential Euler spline $S_n(x;t)$ of degree n to the base t, and also sketches in §7 their role in cardinal spline interpolation. It also presents two new items: 1. In §3 a simplified derivation of the recursive relation

$$S_n(x;t) = \int_x^{x+1} S_{n-1}(u;t)du / \int_0^1 S_{n-1}(u;t)du \quad (n \geq 2)$$

is given, a relation already discussed in Reference [10]. 2. In §6 the approximation of the exponential function 2^x by $S_n(x;t)$ is made more effective by a preliminary subdivision of the interval $[0,1]$ into 2^r parts. By this device our approximation becomes competitive with the modern approximations of 2^x in $[0,1]$ by rational functions. The paper has two aims: 1. As a tribute to Euler, 2. To make the exponential Euler splines $S_n(x;t)$ better known.

AMS (MOS) Subject Classifications: 01A50, 41A15

Key Words: Exponential Euler splines; Euler's bicentenial of 1983

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SIGNIFICANCE AND EXPLANATION

This paper is the author's contribution to the volume "Leonardt Euler - Gedenkband 1983" to be published in Basel, Switzerland, in 1983 in honor of Euler's bicentenial. It has two aims: 1. As a tribute to Euler, 2. To point out Euler's contributions to the subjects of the title.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

EULER'S CONTRIBUTION TO CARDINAL SPLINE INTERPOLATION:
THE EXPONENTIAL EULER SPLINES

I. J. Schoenberg

Introduction. In my monograph [8] of 1973, dedicated to Euler, I already discussed the subjects of the title. On the occasion of the bicentenial of Leonardt Euler we present here an outline of these results, which seem to fit well in what we think of as Eulerian Mathematics.

Our main subject are the exponential Euler splines. In §1 we define them, and §2 shows their close connection with the Eulerian polynomials. In §3 we derive in a simpler way a recursive construction already described in [10]. §§4 and 5 show that the exponential Euler splines of base t converge to the exponential function t^x as their degree tends to infinity. §6 presents an application to the computation of $f(x) = 2^x$. Finally, in §7 we sketch the role of the exponential splines in the problem of cardinal spline interpolation.

1. The exponential Euler splines. We need a few definitions. Let $\mathcal{S}_n = \{S_n(x)\}$ denote the class of cardinal splines $S_n(x)$ of degree n (≥ 1). This means that $S_n(x)$ reduces to a polynomial of degree $\leq n$ in each unit interval $(\nu, \nu+1)$ $(\nu \in \mathbb{Z})$, with the strong restriction that

$$S_n(x) \in C^{n-1}(\mathbb{R}) \quad . \tag{1.1}$$

In particular $S_1(x) \in \mathcal{S}_1$ means that $S_1(x)$ is a continuous piecewise linear function with possible vertices (or "knots") at the integers. Early in this century it was found convenient to represent $S_1(x)$ as a linear combination of shifted versions of the "roof-function"

$$Q_2(x) = x \text{ in } [0,1], = 2 - x \text{ in } [1,2], = 0 \text{ elsewhere} \quad ,$$

so that

$$S_1(x) = \sum_{-\infty}^{\infty} c_j Q_2(x - j)$$

represents uniquely every element of $\mathcal{S}_1$.

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This extends to the class S_n in terms of the forward B-spline

$$(1.2) \qquad Q_{n+1}(x) = \frac{1}{n!}\sum_{\nu=0}^{n+1} (-1)^{\nu}\binom{n+1}{\nu}(x-\nu)_+^n ,$$

where $u_+ = \max(0,u)$. Like $Q_2(x)$, the B-spline $Q_{n+1}(x)$ has remarkable properties:

$$(1.3) \qquad Q_{n+1}(x) > 0 \text{ in } (0,n+1), = 0 \text{ outside } (0,n+1) .$$

Moreover, it is bell-shaped in $(0,n+1)$ and symmetric in its midpoint, i.e.

$$(1.4) \qquad Q_{n+1}(n+1-x) = Q_{n+1}(x) .$$

Clearly $Q_{n+1}(x-j) \in S_n$ for all integers j, and these are the elements of S_n of least support. Again, every element $S(x) \in S_n$ admits a unique "standard" representation of the form

$$(1.5) \qquad S(x) = \sum_{-\infty}^{\infty} c_j Q_{n+1}(x-j) .$$

Definition 1: The exponential splines of base t. Let t be a real or complex number $\neq 0$, and let

$$(1.6) \qquad \Phi_n(x;t) = \sum_{-\infty}^{\infty} t^j Q_{n+1}(x-j) .$$

We call this function the exponential spline of degree n and base t.

Clearly

$$\Phi_n(x+1;t) = \Sigma\, t^j Q_{n+1}(x+1-j) = \Sigma\, t^{j+1} Q_{n+1}(x-j) = t\,\Phi_n(x;t) .$$

Using the representation (1.5) and its unicity, it is easily shown ([7, Lemma 2]) that the most general solution of the functional equation

$$(1.7) \qquad S(x+1) = t\,S(x), \text{ where } S(x) \in S_n$$

is given by

$$(1.8) \qquad S(x) = C\cdot\Phi_n(x;t), \quad (C \text{ is a constant}) .$$

If

$$(1.9) \qquad t = |t|e^{i\alpha}, \ -\pi < \alpha \leq \pi,\ t \neq 0 ,$$

let us try to interpolate the exponential function

$$(1.10) \qquad t^x = |t|^x e^{i\alpha x}$$

at the integers by the function (1.8) so that

$$(1.11) \qquad S(\nu) = t^{\nu} \text{ for all integers } \nu .$$

Because of (1.7) it suffices to determine the constant C in (1.8) so that $S(0) = 1$. The answer is clearly

(1.12) $$S(x) = \frac{\Phi_n(x;t)}{\Phi_n(0;t)} ,$$

but this is possible if and only if $\Phi_n(0;t) \neq 0$. When this holds is easily decided, for by (1.6) and (1.4) we have

$$\Phi_n(0;t) = \Sigma\, t^j Q_{n+1}(-j) = \Sigma\, t^j Q_{n+1}(n+1+j) ,$$

and setting $n+j = \nu$ we find that

(1.13) $$\Phi_n(0;t) = \sum_\nu t^{\nu-n} Q_{n+1}(\nu+1) = t^{-n} \sum_{\nu=0}^{n-1} Q_{n+1}(\nu+1) t^\nu .$$

The result: <u>The interpolation</u> (1.11) <u>with</u> $S(x)$ <u>of the form</u> (1.8) <u>is possible if and only if</u>

(1.14) $$\Pi_n(t) = n! \sum_0^{n-1} Q_{n+1}(j+1) t^j \neq 0 .$$

The polynomial $\Pi_n(t)$ defined by (1.14) is called the <u>Euler-Frobenius polynomial</u>. It is a reciprocal monic polynomial having integer coefficients and having only negative and simple zeros λ_i:

(1.15) $$\lambda_{n-1} < \lambda_{n-2} < \cdots < \lambda_2 < \lambda_1 \; (< 0) .$$

<u>Definition</u> 2: <u>The exponential Euler splines</u> $S_n(x;t)$. <u>Assuming that</u> $\Pi_n(t) \neq 0$, <u>hence that</u>

(1.16) $$\Phi_n(0;t) \neq 0 ,$$

<u>we define</u>

(1.17) $$S_n(x;t) = \frac{\Phi_n(x;t)}{\Phi_n(0;t)} .$$

To summarize $S_n(x;t)$ is the unique cardinal spline interpolant of the exponential t^x satisfying the functional equation

(1.18) $$S_n(x+1;t) = t\, S_n(x;t), \quad (x \in \mathbb{R}) .$$

2. <u>The construction of</u> $S_n(x;t)$ <u>in terms of Eulerian polynomials</u>. How do we construct $S_n(x;t)$? Clearly, its expression by (1.17) is too laborious. This is where Euler comes in. Following Euler, we define the $a_n(t)$ by the expansion

$$\frac{t-1}{t-e^z} = \sum_0^\infty \frac{a_n(t)}{n!} z^n \tag{2.1}$$

The $a_n(t)$ are rational functions of the form

$$a_n(t) = \frac{\Pi_n(t)}{(t-1)^n} , \tag{2.2}$$

where $\Pi_n(t)$ are the polynomials (1.14). For a proof see [7, Lemma 7 on page 391]. The $\Pi_n(t)$ may also be defined by Euler's expansions

$$\frac{\Pi_n(t)}{(1-t)^{n+1}} = \sum_{\nu=0}^\infty (\nu+1)^n t^\nu . \tag{2.3}$$

We find that

$$\Pi_0(t) = \Pi_1(t) = 1,\ \Pi_2(t) = t + 1,\ \Pi_3(t) = t^2 + 4t + 1 ,$$

$$\Pi_4(t) = t^3 + 11t^2 + 11t + 1,\ \Pi_5(t) = t^4 + 26t^3 + 66t^2 + 26t + 1 .$$

On multiplying (2.1) by e^{xz} we obtain Euler's generating function

$$\frac{t-1}{t-e^z} e^{xz} = \sum_0^\infty \frac{A_n(x;t)}{n!} z^n \tag{2.4}$$

of the <u>exponential Euler polynomials</u>

$$A_n(x;t) = x^n + \binom{n}{1}a_1(t)x^{n-1} + \binom{n}{2}a_2(t)x^{n-2} + \cdots + a_n(t) , \tag{2.5}$$

which evidently form an Appell sequence (see [2, Chap. VII, 178]). L. Carlitz [1] writes $A_n(x;t) = H_n(x|t)$ and calls them <u>Eulerian polynomials</u>. See also [1] for extensive references.

The coefficients $a_n(t)$ admit a recursive computation: Multiplying (2.1) by $t-e^z$ we obtain

$$t - 1 = \sum_0^\infty \frac{ta_n(t)}{n!} z^n - \sum_0^\infty \frac{1 + \binom{n}{1}a_1(t) + \cdots + a_n(t)}{n!} z^n$$

and by identifying coefficients of z^n we obtain

$$(2.6) \qquad 1 + \binom{n}{1}a_1(t) + \binom{n}{2}a_2(t) + \cdots + a_n(t) = t\, a_n(t), \quad (n = 1,2,\ldots) \quad ,$$

which show that

$$(2.7) \qquad a_n(t) = \frac{1}{t-1}\left\{1 + \binom{n}{1}a_1(t) + \cdots + \binom{n}{n-1}a_{n-1}(t)\right\}, \ (n = 1,2,\ldots) \quad .$$

Let us remember that we wish to construct $S_n(x;t)$, and that we may exclude the trivial case when $t = 1$, because evidently $S_n(x;1) = 1$ for all x. We ask: <u>What can we say about the function</u> $F(x)$ <u>defined by</u>

$$(2.8) \qquad F(x) = A_n(x;t) \ \underline{\text{if}} \ 0 \leq x < 1 \quad ,$$

<u>and satisfying</u>

$$(2.9) \qquad F(x+1) = t\, F(x) \ \underline{\text{for all}} \ x \ ?$$

We claim that

$$(2.10) \qquad F(x) \in C^{n-1}(\mathbb{R}) \quad .$$

Indeed, from (2.10), and using (2.9) and (2.8), we obtain by differentiation of (2.9) and setting $x = 0$, that we must have that

$$(2.11) \qquad A_n^{(\nu)}(1;t) = t\, A_n^{(\nu)}(0;t), \ (\nu = 1,\ldots,n-1) \ \text{ for } \ n \geq 1 \quad .$$

However, these relations, together with the fact that $A_n(x;t)$ is monic, are known characteristic properties of $A_n(x;t)$, which are derived from (2.4) by ν differentiations with respect to x, subsequently setting $x = 0$ and $x = 1$ in the result. By our result (1.8) concerning (1.7), it follows that

$$(2.12) \qquad F(x) = C\, \Phi_n(x;t) \quad .$$

Assuming (1.14), this proves that $S_n(x;t) = A_n(x;t)/A_n(0;t)$ in $[0,1]$, hence that

$$(2.13) \qquad S_n(x;t) = \{x^n + \binom{n}{1}a_1(t)x^{n-1} + \cdots + a_n(t)\}/a_n(t), \ \text{ if } \ 0 \leq x \leq 1 \quad .$$

<u>Remarks</u>. 1. In [1, page 256, (4.5)] Carlitz already defined the cardinal spline $F(x)$ satisfying (2.8) and (2.9).

2. The continuity requirement (2.11) has been recently stated as a *general principle* concerning the solutions of certain functional equations in the paper [3].

3. <u>A recursive construction of the exponential Euler spline</u> $S_n(x;t)$. This is the subject of my recent paper [10], with the modification that there the sequence

$$(3.1)\qquad S_1(x;t),\ \frac{S_2(x+\frac{1}{2};t)}{S_2(\frac{1}{2};t)},\ S_3(x;t),\ \frac{S_4(x+\frac{1}{2};t)}{S_4(\frac{1}{2};t)},\ \ldots$$

is recursively constructed. <u>We assume</u> t <u>to be non-negative</u>, $t \neq 0$. For the B-spline

$$(3.2)\qquad Q_n(x) = \frac{1}{(n-1)!}\sum_0^n (-1)^\nu \binom{n}{\nu} (x-\nu)_+^{n-1}$$

we easily verify by integration and summation by parts the relation

$$(3.3)\qquad \int_x^{x+1} Q_n(u)\,du = Q_{n+1}(x+1) \quad .$$

For the exponential spline (1.6) this implies that

$$\int_x^{x+1} \Phi_{n-1}(u;t)\,du = \sum_j t^j \int_x^{x+1} Q_n(u-j)\,du = \sum_j t^j \int_{x-j}^{x-j+1} Q_n(u)\,du$$

$$= \sum_j t^j Q_{n+1}(x-j+1) = t\sum_j t^j Q_{n+1}(x-j)$$

and therefore

$$(3.4)\qquad \int_x^{x+1} \Phi_{n-1}(u;t)\,du = t\,\Phi_n(x;t) \quad .$$

For the exponential Euler spline this implies our first proposition

I. <u>If</u> t <u>is not negative</u>, <u>then</u>

$$(3.5)\qquad S_n(x;t) = \int_x^{x+1} S_{n-1}(u;t)\,du \Big/ \int_0^1 S_{n-1}(u;t)\,du, \quad (n = 2,3,\ldots) \quad .$$

This is remarkable recursive construction: Starting from the linear Euler spline $S_1(x;t)$, (3.5) recursively furnishes all higher degree $S_n(x;t)$. Also notice that (3.5) does not depend on t explicitly.

We also need the following result established in [8, Lecutre 2, §5]:

II. <u>If</u>

$$(3.6)\qquad t = |t|e^{i\alpha},\ -\pi < \alpha < \pi,\ t \neq 0,\ t \neq 1 \quad ,$$

<u>which implies that negative values of</u> t <u>are excluded</u>, <u>then</u>

$$(3.7)\qquad \Phi_n(x;t) \neq 0 \quad \text{for all real } x \quad .$$

The reason given in [8, loc.cit.] is as follows: 1. If $t > 0$, then the curve of the complex plane

(3.8) $$\Gamma : z = \Phi_n(x;t) \quad (-\infty < x < +\infty)$$

is clearly contained within the positive half of the real axis. 2. If in (3.6) we have $0 < \alpha < \pi$, say, then the curve (3.8) spirals convexly about the origin, never assuming the value $z = 0$. This is shown by induction in n.

4. *A series expansion of* $S_n(x;t)$. Again we assume that (3.6) holds. Already in [7, §7] we derived by means of residue theory the following proposition

III. *Let*

(4.1) $$\gamma = \log t = \log |t| + i\alpha, \quad \textit{hence} \quad t = e^{\gamma} \; ,$$

and let (3.6) *hold*. *Then*

(4.2) $$S_n(x;t) = \sum_{-\infty}^{\infty} \frac{1}{(\gamma + 2\pi ik)^{n+1}} e^{(\gamma+2\pi ik)x} \Big/ \sum_{-\infty}^{\infty} \frac{1}{(\gamma+2\pi ik)^{n+1}} \; .$$

An alternative derivation of (4.2) uses the recursive relation (3.5) and proceeds as follows. To simplify notations we write

(4.3) $$S_n(x) = S_n(x;t) \; .$$

Observe that $S_1(x)$ is the linear spline that interpolates the sequence (t^k), and so

$$S_1(x) = 1 + (t-1)x \quad \text{if} \quad 0 \leq x \leq 1 \; .$$

However $S_1(x)t^{-x}$ is periodic with period 1, because $S_1(x+1)t^{-x-1} = tS_1(x)\, t^{-x-1} = S_1(x)t^{-x} = S_1(x)e^{-\gamma x}$. Let its Fourier series be $S_1(x)e^{-\gamma x} = \sum_k a_k\, e^{2\pi ikx}$. For its coefficients we find by an integration by parts (see [10, §5]) that

$$a_k = \frac{(t-1)^2}{2\pi t} \frac{1}{(\gamma+2\pi ik)^2}$$

and therefore

(4.4) $$S_1(x) = \frac{(t-1)^2}{2\pi t} \sum_{-\infty}^{\infty} \frac{e^{(\gamma+2\pi ik)x}}{(\gamma+2\pi ik)^2} \; .$$

Since $S_1(x)$ is up to a non-vanishing factor identical with $\Phi_1(x;t)$, it suffices, by (3.5), to perform the operation $\int_x^{x+1}(\cdot)du$ on $S_1(x)$ a total of $n-1$ times, and to divide the final result by its value at $x=0$. However

(4.5) $$\int_x^{x+1} e^{(\gamma+2\pi ik)u}du = \frac{e^{\gamma}-1}{\gamma+2\pi ik} e^{(\gamma+2\pi ik)x} \; .$$

Performing the operations as described on (4.4), we obtain the fraction (4.2).

5. $S_n(x;t) \to t^x$ as $n \to \infty$ for non-negative t. Let us assume (3.6), so that t is non-negative, and write (4.2) as

$$(5.1) \qquad S_n(x;t)t^{-x} = \sum_{-\infty}^{\infty} \frac{1}{(\gamma+2\pi ik)^{n+1}} e^{2\pi ikx} \Big/ \sum_{-\infty}^{\infty} \frac{1}{(\gamma+2\pi ik)^{n+1}} .$$

We multiply each of the two series of this fraction by γ^{n+1}; except for the two terms for $k = 0$, which are $= 1$ in both series, the k-th terms in both series are in absolute value $= |\gamma/(\gamma+2\pi ik)|^{n+1}$. Writing

$$(5.2) \qquad \rho = \log|t|, \quad \text{hence} \quad \gamma = \log t = \rho + i\alpha \quad ,$$

we find that

$$|\gamma/(\gamma+2\pi ik)|^2 = |(\rho + i\alpha)|^2/|\rho + i(\alpha+2\pi ik)|^2 = \frac{\rho^2+\alpha^2}{\rho^2+(\alpha+2\pi k)^2} .$$

From $-\pi < \alpha < \pi$ we find that

$$(5.3) \qquad |\alpha + 2\pi k| \geq 2\pi - |\alpha| > |\alpha|$$

and therefore

$$(5.4) \qquad \max_k |\gamma/(\gamma+2\pi ik)| = \left(\frac{\rho^2+\alpha^2}{\rho^2+(2\pi-|\alpha|)^2}\right)^{1/2} = \delta_t < 1$$

by (5.3). Moreover $\delta_t > 0$, because ρ and α can not both vanish, as we assume that $t \neq 1$.

Now it should be clear that the right side of (5.1) is $= 1 + O(\delta_t^{n+1})$ and that we may write

$$(5.5) \qquad \frac{S_n(x;t)}{t^x} - 1 = O(\delta_t^{n+1}) \quad \text{uniformly for } x \in \mathbb{R} .$$

Notice that the approximation (5.5) deteriorates as α approaches $\pm\pi$ because δ_t approaches 1. As the constant in front of the O-term of (5.5) depends on t, but not on x and not on n, we have established the proposition

IV. If t is not negative, then

$$(5.6) \qquad \lim_{n\to\infty} S_n(x;t) = t^x = |t|^x e^{i\alpha x} \quad \text{for all real } x .$$

6. <u>The computation of the exponential function</u> $f(x) = 2^x$. Can the approximation (2.13), for $t = 2$, be used to compute 2^x in view of the convergence theorem (5.6)? In [7, §11], and again in [10, §5] I stated that this seems practicable. Actually we find that this method does not compete in accuracy with the modern approximations of 2^x in [0,1] by appropriate <u>rational functions</u> (see [5]). However, we will show that by appropriate binary subdivisions of [0,1] our approach becomes competitive.

We introduce the natural number r and change variables by

$$x = \frac{1}{2^r} z \quad , \tag{6.1}$$

defining $F(z)$ by

$$F(z) = 2^{z/2^r} = 2^x = f(x) \quad . \tag{6.2}$$

For the base

$$t = 2^{1/2^r} \tag{6.3}$$

we have $F(z) = t^z$, and this we can approximate, in view of (5.6), by

$$F(z) \approx S_n(z; 2^{1/2^r}) \quad . \tag{6.4}$$

Setting

$$z = 2^r x = [2^r x] + \theta = \nu + \theta, \quad (0 \leq \theta < 1) \quad , \tag{6.5}$$

where $[\cdot]$ has its usual meaning, and

$$\nu = [2^r x] \quad . \tag{6.6}$$

However

$$S_n(z;2^{1/2^r}) = S_n(\nu+\theta;t) = t^\nu S_n(\theta;t) = 2^{\nu/2^r} S_n(\theta;2^{1/2^r}) \quad , \tag{6.7}$$

and by (6.2) and (6.4) we have

$$2^x \approx 2^{\nu/2^r} S_n(\theta;2^{1/2^r}) \quad . \tag{6.8}$$

We recall that by (2.13) we have

$$S_n(\theta;t) = P_{n,r}(\theta), \quad (0 \leq \theta < 1) \quad , \tag{6.9}$$

where $P_{n,r}(u)$ is the polynomial

$$\begin{aligned} P_{n,r}(u) &= \{u^n + \tbinom{n}{1} a_1(t) u^{n-1} + \cdots + a_n(t)\}/a_n(t) \\ &= 1 + c_{1,r}^{(n)} u + \cdots + c_{n,r}^{(n)} u^n \quad , \end{aligned} \tag{6.10}$$

whose coefficients are compute by Euler's algorithm (2.7).

How close does $S_n(\theta;t)$ approximate t^θ? In (5.4) we have, by (6.3) and (5.2), that $\alpha = 0$ and $\rho = \log|t| = (\log 2)/2^r$, and so, dropping the term = 1 in the denominator, we have

$$\delta_{t,r} = \left(\frac{1}{1 + \left(\frac{2^r 2\pi}{\log 2}\right)^2}\right)^{1/2} < \frac{\log 2}{2\pi 2^r} \tag{6.11}$$

and (5.5) shows that

$$\frac{S_n(\theta;t)}{t^\theta} - 1 = O\left(\frac{\log 2}{2\pi 2^r}\right)^{n+1} . \tag{6.12}$$

We conclude that the approximation (6.8) will be close, provided that either n or r, or both, are of some size.

We need the numerical values of the coefficients of the polynomial (6.10). I am indebted to Fred Sauer, of the MRC Computing Staff, for a 30 place table of the coefficients

$$c_{i,r}^{(n)} \quad \text{for} \quad i = 1,\ldots,n; \quad n = 3,\ldots,20; \quad r = 0,1,\ldots,16 \quad . \tag{6.13}$$

Computing with a hand-held calculator, we record here only their values for $r = 4$ and $n = 3$ to 12 decimal places:

$$c_{1,4}^{(3)} = .04332\ 16979\ 37, \quad c_{2,4}^{(3)} = .00093\ 82380\ 41, \quad c_{3,4}^{(3)} = .00001\ 38464\ 49 \quad . \tag{6.14}$$

These should give in (6.8) about eight decimal places of accuracy.

As an example we compute $f(\pi) = 2^\pi$. Here $x = \pi$, $z = 2^r x = 16\pi = 50.26548\ 246$, and so $\nu = 50$, $\theta = .26548\ 24574$. Since $50/16 = (2^5 + 2^4 + 2)/2^4 = 2 + 1 + 2^{-3} = 3 + 1/8$, (6.8) becomes

$$2^\pi = 2^{50/16}\ F(\theta) \approx 8\cdot 2^{1/8}\ P_{3,4}(\theta) \quad .$$

We find from (6.14) that $P_{3,4}(\theta) = 1.01156\ 7538$, and finally

$$2^\pi \approx 8.82497\ 7778$$

which is accurate to seven decimals. In this computation we have approximated 2^x by the interpolating cubic spline $S_n(16x;2^{1/16})$ having $2^4 = 16$ components in the interval $[0,1]$. In implementing (6.8) it is important to represent the integer $\nu = [2^r x]$ in binary notation, as we have done above. In this way $2^{\nu/2^r}$ appears as a polynomial in

$2^{1/2^r}$ having only coefficients 0 or 1. For $r = 4$ and $n = 9$ Sauer's table of (6.13) gives a result accurate to 21 decimals.

7. Cardinal spline interpolation. In this last section we sketch a solution of the problem of cardinal spline interpolation [8, Lecture 4]:

The (y_j) *being prescribed, we wish to find*

(7.1) $$S(x) \in \mathcal{S}_n$$

such that

(7.2) $$S(j) = y_j \text{ for all integers } j .$$

This problem is trivial if $n = 1$ and we may assume $n \geq 2$. We further restrict the discussion to the case when

(7.3) $$\text{the data } (y_j) \text{ and the solution } S(x) \text{ are of power growth} ,$$

meaning that $|y_j| = 0|j|^{\gamma}$ as $j \to \pm\infty$ for some $\gamma \geq 0$, and that $|S(x)| = 0|x|^{\delta}$ as $x \to \pm\infty$, for some $\delta \geq 0$. We shall deal only with the *question of uniqueness* of the solution, because the exponential splines $\Phi_n(x;t)$ play a decisive role.

We need the *null-space*

(7.4) $$\mathcal{S}_n^0 = \{S(x);\ S(x) \in \mathcal{S}_n,\ S(j) = 0 \text{ for all integers } j\}$$

and state

Lemma 1. *The linear subspace* $\mathcal{S}_n^0$, *of* $\mathcal{S}_n$, *has the dimension* $n - 1$ *and is spanned by the* $n - 1$ *exponential splines*

(7.5) $$S_\nu(x) = \Phi_n(x;\lambda_\nu) \quad (\nu = 1,\ldots,n-1) ,$$

where the λ_ν *are the zeros* (1.15) *of the polynomial*

(7.6) $$\Pi_n(t) = n! \sum_0^{n-1} Q_{n+1}(j+1)t^j .$$

That the $S_\nu(x)$, called the *eigensplines of* $\mathcal{S}_n$, are elements of $\mathcal{S}_n^0$ follows from (1.13) and the functional equations

(7.7) $$S_\nu(x+1) = \lambda_\nu S_\nu(x), \quad (\nu = 1,\ldots,n-1) .$$

Indeed, $S_\nu(0) = 0$ implies that $S_\nu(j) = 0$ for all j. For a proof of Lemma 1 see [8, Lecture 4, §3].

The polynomial $\Pi_n(t)$ being reciprocal, we have

(7.8) $$\lambda_{n-1}\lambda_1 = \lambda_{n-2}\lambda_2 = \cdots = 1$$

and there is an important distinction depending on the parity of n.

1. $n = 2m-1$ is odd. From the simplicity of the λ_ν it follows that

(7.9) $$\lambda_{2m-2} < \cdots < \lambda_m < -1 < \lambda_{m-1} < \cdots < \lambda_1 < 0 \quad .$$

By Lemma 1 $S(x) \in {}^0_n$ implies that

(7.10) $$S(x) = \sum_1^{n-1} c_\nu S_\nu(x) \quad .$$

The inequalities (7.9) and the behavior of the splines (7.5) at $+\infty$ and $-\infty$ implies that (7.10) is of power growth if and only if all $c_\nu = 0$. This is the basis of our first result

V. If $n = 2m-1$ is odd, and the (y_j) are of power growth, then there exists a unique $S(x) \in \mathcal{S}_n$ of power growth satisfying (7.2).

2. $n = 2m$ is even. Now we find by (7.8) that the λ_ν satisfy

(7.11) $$\lambda_{2m-1} < \cdots < \lambda_{m+1} < \lambda_m = -1 < \lambda_{m-1} < \cdots < \lambda_1 < 0 \quad ,$$

so that

(7.12) $$S_m(x) = \Phi_n(x;-1)$$

is one of the eigensplines. It satisfies $S_m(x+1) = -S_m(x)$, hence has period $= 2$; Nörlund [6, Chap. 2, §16] attributes it to Hermite and Sonine. The bounded $S_m(x)$ is a counter-example to the proposition V for even n.

We now abandon the class $\mathcal{S}_n$ and consider the new class

(7.13) $$\mathcal{S}_n^* = \{S(x);\ S(x+\tfrac{1}{2}) \in \mathcal{S}_n\} \quad .$$

This is the class of midpoint cardinal splines having their knots at $x = j + \frac{1}{2}$. Within $\mathcal{S}_n^*$ we have the midpoint exponential splines defined by

(7.14) $$\Phi_n^*(x;t) = \Phi_n(x+\tfrac{1}{2};\ t) \quad .$$

When are these element of the null-space

(7.15) $$\mathcal{S}_n^{*0} = \{S(x);\ S(x) \in \mathcal{S}_n^*;\ S(j) = 0 \ \text{ for all } j\}\ ?$$

This depends on the vanishing of

$$\Phi_n^*(0;t) = \Phi_n(\tfrac{1}{2};t) = \Sigma\, t^j Q_{n+1}(\tfrac{1}{2} - j) = \Sigma\, t^j Q_{n+1}(n+1-\tfrac{1}{2}+j)$$

$$= \Sigma\, t^{j-n} Q_{n+1}(j+\tfrac{1}{2}) = t^{-n} \sum_0^n Q_{n+1}(j+\tfrac{1}{2}) t^j$$

which follows from (1.3). This shows that the eigensplines of S_n^* depend on the zeros of the polynomial

$$\Pi_n^*(t) = 2^n\, n! \sum_0^n Q_{n+1}(j+\tfrac{1}{2}) t^j \quad , \tag{7.16}$$

which is called the <u>midpoint Euler-Frobenius polynomial</u>. Again it is monic and reciprocal having n simple and negative zeros μ_ν:

$$\mu_n < \mu_{n-1} < \cdots < \mu_2 < \mu_1 < 0 \quad . \tag{7.17}$$

The analogue of Lemma 1 is

<u>Lemma</u> 2. <u>The linear subspace</u> S_n^{*0}, <u>of</u> S_n^*, <u>is of dimension</u> n <u>and it is spanned by the</u> n <u>midpoint exponential splines</u>

$$S_\nu^*(x) = \Phi_n^*(x;\mu_\nu) = \Phi_n(x+\tfrac{1}{2};\mu_\nu), \quad (\nu = 1,\ldots,n) \quad . \tag{7.18}$$

It follows that the general element of S_n^{*0} is

$$S(x) = \sum_1^n c_\nu S_\nu^*(x) \quad . \tag{7.19}$$

The role of the parity of n is now reversed, because again we have

$$\mu_n\mu_1 = \mu_{n-1}\mu_2 = \cdots = 1 \quad . \tag{7.20}$$

1. $n = 2m-1$ <u>is odd</u>. Now (7.17) and (7.20) show that

$$\mu_{2m-1} < \cdots < \mu_{m+1} < \mu_m = -1 < \mu_{m-1} < \cdots < \mu_1 < 0 \quad , \tag{7.21}$$

so that S_n^{*0} contains the periodic, hence <u>bounded</u>, element

$$S_m^*(x) = \Phi_n(x+\tfrac{1}{2}; -1) \tag{7.22}$$

and there can be <u>no unicity</u> for the problem (7.2) within the subclass of S_n^* of power growth.

2. $n = 2m$ <u>is even</u>. Now (7.17) becomes

$$\mu_{2m} < \cdots < \mu_{m+1} < -1 < \mu_m < \cdots < \mu_1 < 0 \tag{7.23}$$

and these inequalities allow us to show that if (7.19) is of power growth, then all $c_\nu = 0$. These results lead to the analogue of proposition V:

VI. <u>If</u> $n = 2m$ <u>is even, and</u> (y_j) <u>are of power growth, then there exists a unique</u> $S(x) \in \mathcal{S}_n^*$ <u>of power growth satisfying</u> (7.2).

<u>Concluding remarks</u>. 1. The special case when the data (y_j) are <u>bounded</u>, and the solution $S(x)$ is to be <u>bounded</u>, the propositions V and VI were first established by Subbotin [11]. 2. If in (7.2) we have $y_j = (-1)^j$ for all j, then the solution of (7.2) is the function

$$(7.24) \qquad \mathcal{E}_n(x) = \begin{cases} \Phi_n(x;-1)/\Phi_n(0;-1) & \text{if } n \text{ is odd}, \\ \Phi_n(x+\frac{1}{2};-1)/\Phi_n(\frac{1}{2};-1) & \text{if } n \text{ is even}. \end{cases}$$

It is called <u>the Euler spline of degree</u> n, and it is the solution of the famous Landau-Kolmogorov extremum problem (see [9] for references).

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This paper is the author's contribution to the volume "Leonardt Euler-Gedenkband 1983" to be published in Basel, Switzerland, in 1983 in honor of Euler's bicentenial. It is mainly devoted to the exponential Euler spline $S_n(x;t)$ of degree n to the base t, and also sketches in §7 their role in cardinal spline interpolation. It also presents two new items: 1. In §3 a simplified derivation of the recursive relation

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ABSTRACT (continued)

$$S_n(x;t) = \int_x^{x+1} S_{n-1}(u;t)du \Big/ \int_0^1 S_{n-1}(u;t)du \quad (n \geq 2)$$

is given, a relation already discussed in Reference [10]. 2. In §6 the approximation of the exponential function 2^x by $S_n(x;t)$ is made more effective by a preliminary subdivision of the interval $[0,1]$ into 2^r parts. By this device our approximation becomes competitive with the modern approximations of 2^x in $[0,1]$ by rational functions. The paper has two aims: 1. As a tribute to Euler, 2. To make the exponential Euler splines $S_n(x;t)$ better known.

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